Total number of printed pages–8

2 SEM PG (CBCS) MTH C 5

2025

(June)

MATHEMATICS

Paper : MTH C 5

(Linear Algebra)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

UNIT–I

1. Answer the following questions : $1 \times 2 = 2$

(a) Is $\left\{(a, b, c) \in \mathbb{R}^3 \mid a^3 = b^3\right\}$ a subspace of $\mathbb{R}^3$?

(b) *True* **or** *False* : "The subspaces of $\mathbb{R}^2$ are precisely $\{0\}$, $\mathbb{R}^2$ and all lines in $\mathbb{R}^2$ containing the orgin."

2. Answer ***any two*** of the following questions : 4×2=8

(a) Suppose $\{v_1, v_2, v_3, v_4\}$ is a basis of V. Prove that $\{v_1+v_2, v_2+v_3, v_3+v_4, v_4\}$ is also a basis of V.

(b) Suppose $P_4(\mathbb{R})$ is the the vector space containing all the real polynomials of degree 4 and let
$U = \{p \in P_4(\mathbb{R}) \mid p(2) = p(5) = p(6)\}$.
Find a basis of U. Also find a subspace W of $P_4(\mathbb{R})$ such that $P_4(\mathbb{R}) = U \oplus W$.

(c) *Prove* **or** *disprove* : If V_1, V_2, U are subspaces of V such that $V = V_1 \oplus U$ and $V = V_2 \oplus U$, then $V_1 = V_2$.

UNIT–II

3. Answer the following questions : 1×2=2

(a) *True* **or** *False* : "If a linear map takes a non-zero vector to 0, then the map is not invertible."

(b) *True* **or** *False* : "Suppose V and W are finite dimensional vector spaces such that $dim(V) > dim(W)$. Then no linear map from V to W is injective."

4. Answer ***any two*** of the following questions : 4×2=8

(a) Let T be a linear map from $\mathbb{R}^4$ to $\mathbb{R}^2$ such that

$$null(T) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 = 5x_2 \text{ and } x_3 = 7x_4\}.$$

Prove that T is surjective.

(b) Give an example of a linear map which is injective but not invertible.

(c) Prove that a map T is invertible *iff* 0 is not an eigenvalue of T.

(d) Let V is finite dimensional vector space, $T \in \mathcal{L}(V, W)$ and U is a subspace of W. Then show that $\{v \in V \mid T(v) \in U\}$ is a subspace of V. Also show that

$$dim(\{v \in V \mid T(v) \in U\}) = dim(null(T)) + dim(U \cap \text{range } T).$$

UNIT–III

5. Answer ***any three*** of the following questions : 4×3=12

(a) Suppose $T \in \mathcal{L}(V)$. Then prove that every list of eigenvectors of T corresponding to distinct eigenvalues of T is linearly independent.

(b) Suppose $P(\mathbb{R})$ is the vector space of all real polynomials and let $T \in \mathcal{L}(P(\mathbb{R}))$ be defined by $T(p) = p'$. Find all eigenvalues and eigenvectors of T.

(c) Suppose V is a finite dimensional vector space and $T \in \mathcal{L}(V)$. Then show that for a scaler λ, the following statements are equivalent :

(i) λ is an eigenvalue of T.

(ii) $T - \lambda I$ is not injective.

(iii) $T - \lambda I$ is not surjective.

(iv) $T - \lambda I$ is not invertible.

(d) Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of every collection of subspaces of V invariant under T is invariant under T.

(e) Suppose $T \in \mathcal{L}(V)$ is such that every nonzero vector in V is an eigenvector of T. Prove that T is a scalar multiple of the identity operator.

UNIT–IV

6. Answer the following questions : 2×3=6

(a) Suppose V is a real inner product space. Show that if $u, v \in V$ have the same norm, then $u+v$ is orthogonal to $u-v$.

(b) Suppose $e_1, e_2, \ldots, e_m$ is an orthonormal list of vectors in V. If $v \in V$, then show that

$$|\langle v, e_1\rangle|^2 + |\langle v, e_2\rangle|^2 + \ldots\ldots + |\langle v, e_m\rangle|^2 \le \|v\|^2.$$

(c) Define positive operator. Give a non-trivial example.

7. Answer ***any two*** of the following questions : 4×2=8

(a) Suppose V is a complex inner product space. Prove that

$$\langle u, v\rangle = \left(\|u+v\|^2 - \|u-v\|^2 + \|u+iv\|^2 i - \|u-iv\|^2 i\right)/4$$

for all $u, v \in V$.

(b) Suppose U be a finite dimensional subspace of V. Then prove that $V = U \oplus U^{\perp}$.

(c) Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$. Prove that T is self-adjoint *iff* $\langle T(v), v \rangle \in \mathbb{R}$ for every $v \in V$.

(d) Suppose $T \in \mathcal{L}(V)$ is a normal operator. Then show that

(i) $T - \lambda I$ is also normal for every scaler λ.

(ii) if $v \in V$ and λ is a scaler, then $T(v) = \lambda v$ *iff* $T^*(v) = \bar{\lambda} v$.

UNIT–V

8. Answer the following questions : 2×3=6

(a) Define $T \in \mathcal{L}(\mathbb{C}^2)$ by $T(x, z) = (z, 0)$. Find all the generalized eigenvectors of T.

(b) Suppose $T \in \mathcal{L}(V)$. Prove that if nulity $(T^5) = 8$ and nulity $(T^7) = 9$, then nulity $(T^m) = 9$ for all integers $m \geq 6$.

(c) Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then prove that T is not invertible if and only if the constant term of the minimal polynomial of T is 0.

9. Answer ***any two*** of the following questions : 4×2=8

(a) Let A and B be two matrices in $\mathbb{M}_n(\mathbb{F})$ having the same trace and the same minimal polynomial of degree $n-1$. Prove that the characteristic polynomials of A and B are the same.

(b) *Prove* **or** *disprove* : If the minimal polynomial of a linear operator having characteristic polynomial $x^3(x-1)^2$ is $x(x-1)$, then its nulity is 3.

(c) Find the Jordan form of a matrix $A \in \mathbb{M}_7(\mathbb{R})$ having characteristic polynomial $(x-1)^3(x-2)^4$ with $nulity(A - I_7) = 2$ and $nulity(A - 2I_7) = 3$.

(d) Calculate the bilinear form f_A on $\mathbb{R}^2$ if $A = \begin{pmatrix} 1 & b \\ a & 3 \end{pmatrix}$. Can f_A be skew-symmetric ?