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**3 SEM TDC MTMH (CBCS) C 6**

**2023**

( Nov/Dec )

**MATHEMATICS**

( Core )

Paper : C-6

**( Group Theory—I )**

*Full Marks : 80*

*Pass Marks : 32*

*Time : 3 hours*

*The figures in the margin indicate full marks  
for the questions*

1. (a) In  $GL(2, Z_{11})$ , find  $\det \begin{bmatrix} 9 & 4 \\ 2 & 6 \end{bmatrix}$ . 1
- (b) Show that  $\{1, 2, 3\}$ , under multiplication modulo 4 is not a group. 1
- (c) Show that the inverse of every element in a group is unique. 2

( 2 )

(d) Write out a complete Cayley table for  $D_3$ .  
Is  $D_3$  Abelian? 2+1=3

(e) Let  $G$  be a group and  $a \in G$ . Then  
prove that  $O(a) = O(x^{-1}ax), \forall x \in G$ . 3

(f) Show that the set of six transformations  $f_1, f_2, f_3, f_4, f_5, f_6$  on the set of complex numbers defined by

$$f_1(z) = z, \quad f_2(z) = \frac{1}{z}, \quad f_3(z) = 1 - z$$

$$f_4(z) = \frac{z}{z-1}, \quad f_5(z) = \frac{1}{1-z}, \quad f_6(z) = \frac{z-1}{z}$$

form a finite non-Abelian group under  
composite of functions. 5

2. (a) In the group  $Z$ , find  $\langle 12, 18, 45 \rangle$ . 1

(b) Let  $G$  be a group and let  $a$  be any  
element of  $G$ . Then prove that  $\langle a \rangle$  is a  
subgroup of  $G$ . 2

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( Continued )

( 3 )

(c) Define product of two subgroups of a  
group and write the condition that the  
product of two subgroups will be a  
subgroup of that group. 1+1=2

(d) In the group  $Z_{12}$ , find  $|a|$  and  $|a+b|$   
if  $a = 6$  and  $b = 2$ . 2

(e) Define normalizer of an element of a  
group and also prove that the  
normalizer  $N(a)$  of  $a \in G$  is a subgroup  
of  $G$ . 1+2=3

(f) Prove that a non-empty subset  $H$  of a  
finite group  $G$  is a subgroup of  $G$  if  
and only if  $HH = H$ . 5

Or

Prove that the union of two subgroups  
of a group is a subgroup of the group if  
and only if one is contained in the  
other.

3. (a) Give an example of a cyclic group  
whose order is not prime. 1

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( Turn Over )

- (b) State Fermat's little theorem. 1
- (c) Express the following permutation as a product of disjoint cycles. Also find whether it is even or odd :  $1+1=2$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 7 & 3 & 1 & 4 & 2 & 6 \end{pmatrix}$$

- (d) Prove that an infinite cyclic group has exactly two generators. 3
- (e) If a permutation  $\alpha$  can be expressed as a product of an even (odd) number of transpositions, then prove that every decomposition of  $\alpha$  into a product of transpositions must have even (odd) number of transpositions. 3
- (f) Prove that the set  $A_n$  of all even permutations on a set  $S$  having  $n \geq 2$  elements is a subgroup of  $S_n$  of order  $\frac{n!}{2}$ . 5

- (g) Let  $G$  be a group and  $H$  be a subgroup of  $G$ . Let  $a, b \in G$ . Then prove that
- (i)  $Ha = Hb$  if and only if  $ab^{-1} \in H$
- (ii)  $Ha$  is a subgroup of  $G$  iff  $a \in H$  5

Or

Let  $G$  be a group and  $H, K$  be two subgroups of  $G$ . Then prove that

$$O(HK) = \frac{O(H)O(K)}{O(H \cap K)}$$

4. (a) What is the order of the element  $(g_1, g_2, \dots, g_n)$  of the external direct product of the groups  $(G_1, G_2, \dots, G_n)$ ? 1
- (b) Prove that the quotient group of an Abelian group is Abelian. 2
- (c) Prove that  $Z_2 \oplus Z_3$  is cyclic. 2
- (d) If  $H$  is a subgroup of a group  $G$  such that  $i_G(H) = 2$ , then prove that  $H$  is normal subgroup in  $G$ . 5

- (e) Let  $G$  be a group and  $Z(G)$  be the centre of  $G$ . If  $\frac{G}{Z(G)}$  is cyclic, then prove that  $G$  is Abelian. 5

Or

Let  $G$  and  $H$  be finite cyclic groups. Then prove that  $G \oplus H$  is cyclic if and only if  $|G|$  and  $|H|$  are relatively prime.

5. (a) Let  $f$  be an isomorphic mapping of a group  $G$  into a group  $G'$ . Then prove that the  $f$ -image of the inverse of an element  $a$  of  $G$  is the inverse of the  $f$ -image of  $a$ . 2

- (b) Let  $G$  be any group and  $a$  be any fixed element in  $G$ . Define a mapping  $f: G \rightarrow G$  by  $f(x) = axa^{-1}, \forall x \in G$ . Then prove that  $f$  is an isomorphism of  $G$  onto itself. 3

- (c) Prove that the relation of isomorphism in the set of all groups is an equivalence relation. 5

( Continued )

- (d) Show that the multiplicative group  $G = \{1, \omega, \omega^2\}$  is isomorphic to the permutation group  $G' = \{I, (abc), (acb)\}$  on three symbols  $a, b, c$ . 5

Or

If  $H$  and  $K$  be two normal subgroups of  $G$  such that  $H \subseteq K$ , then prove that

$$\frac{G}{K} \cong \frac{G/H}{K/H}$$

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