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3 SEM TDC MTMH (CBCS) C 5

2025

(Nov/Dec)

MATHEMATICS

(Core)

Paper : C-5

(Theory of Real Functions)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

*The figures in the margin indicate full marks
for the questions*

1. (a) State True or False : 1
Every point of the interval $[0, 1]$ is a cluster point of the interval $(0, 1)$.
- (b) Define uniform continuity of a function on a set. 1

(2)

(c) Find the cluster points of the set $A = \{1, 2\}$. 2

(d) State the sequential criterion for the limit of a function. 2

(e) If $\text{sgn}(x)$ denotes the signum function of x , then show that

$\lim_{x \rightarrow 0} \text{sgn}(x)$
does not exist. 2

(f) Using definition, show that

$\lim_{x \rightarrow \infty} x^n = \infty$ for $n \in \mathbb{N}$ 2

(g) State the discontinuity criterion for a function at a point. 2

(h) If $f: A \rightarrow \mathbb{R}$ and C is a cluster point of A , then prove that f can have only one limit at C . 3

(i) Use ϵ - δ definition to establish that

$\lim_{x \rightarrow c} \frac{1}{x} = \frac{1}{c}$ 3

(3)

(j) Let I be a closed bounded interval and let $f: I \rightarrow \mathbb{R}$ be continuous on I . Then prove that the set $f(I) = \{f(x) : x \in I\}$ is a closed bounded interval. 3

(k) State and prove preservation interval theorem. 1+3=4

(l) State the squeeze theorem. Applying this theorem, show that

$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ 2+3=5

2. (a) State the interior extremum theorem. 1

(b) Show that the function $g: [-1, 1] \rightarrow \mathbb{R}$ defined by

$$g(x) = \begin{cases} 1 & \text{for } 0 < x \leq 1 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } -1 \leq x < 0 \end{cases}$$

is not the derivative of any function on $[-1, 1]$. 2

(c) Let $I \subseteq \mathbb{R}$ be an interval and $f: I \rightarrow \mathbb{R}$ such that f has derivative at c . If $f'(c) > 0$, then show that there is a number $\delta > 0$ such that

$$f(x) > f(c), \forall x \in (c, c + \delta) \quad 3$$

- (d) Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b) . Show that if

$$\lim_{x \rightarrow a} f'(x) = A$$

then $f'(a)$ exists and equals A . 3

Or

Let I be an interval and let $f : I \rightarrow \mathbb{R}$ be differentiable on I . Show that if the derivative f' is positive on I , then f is strictly increasing on I .

- (e) Let f be a continuous function on the interval $I = [a, b]$ and c be an interior point of I . If f is differentiable on (a, c) and (c, b) and if there is a neighbourhood $(c - \delta, c + \delta) \subseteq I$ such that $f'(x) \geq 0$ for $c - \delta < x < c$ and $f'(x) \leq 0$ for $c < x < c + \delta$, then prove that f has relative maximum at c . 3

- (f) Use the mean value theorem to show that

$$\frac{x-1}{x} < \ln x < x-1 \text{ for } x > 1 \quad 4$$

- (g) If $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = 2x + \frac{1}{x^2}$ for $x \neq 0$, then find the points of relative extrema and the intervals on which the function is increasing. 4
- (h) State and prove Rolle's theorem. 1+4=5

3. (a) What is the necessary condition for the existence of relative maximum of a differentiable function $f : I \rightarrow \mathbb{R}$ at a point $C \in I$? 1

- (b) State True or False : 1
A convex function on an open interval is necessarily continuous.

- (c) Prove that 2
$$1 - \frac{x^2}{2} \leq \cos x$$
for all $x \in \mathbb{R}$.

- (d) Let $I \subseteq \mathbb{R}$ be an open interval and let $f : I \rightarrow \mathbb{R}$ be differentiable on I and $f''(a)$ exists at $a \in I$. Show that

$$f''(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} \quad 3$$

(6)

Or

If $f(x) = e^x$, then show that the remainder term in Taylor's theorem converges to zero as $x \rightarrow \infty$ for each fixed x_0 and x .

- (e) Determine whether or not $x=0$ is a point of relative extremum of the function

$$f(x) = \sin x - x$$

3

- (f) Let I be an open interval and let $f: I \rightarrow \mathbb{R}$ have a second derivative on I . Then prove that f is a convex function on I iff

$$f''(x) \geq 0 \quad \forall x \in I$$

5

- (g) State the Maclaurin's theorem. Stating the condition of validity of the expansion, expand $\cos x$ by using Maclaurin's series.

2+3=5

- (h) State the Taylor's theorem and derive the Cauchy's form of remainder after n terms in Taylor's series.

2+3=5

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(Continued)

(7)

Or

Let I be an interval and x_0 be an interior point of I . For $n \geq 2$, suppose that the derivatives $f', f'', \dots, f^{(n)}$ exist and are continuous in a neighbourhood of x_0 and that $f'(x_0) = \dots = f^{(n-1)}(x_0) = 0$, but $f^{(n)}(x_0) \neq 0$. Then prove that f has a relative minimum or maximum at x_0 according as $f^{(n)}(x_0) > 0$ or $f^{(n)}(x_0) < 0$, when n is even. Also show that if n is odd, then f has neither a relative maximum nor a relative minimum at x_0 .

5

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