3 SEM TDC MTMH (CBCS) C 5

2025

(Nov/Dec)

MATHEMATICS

(Core)

Paper : C–5

(Theory of Real Functions)

Full Marks : 80
Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* State True or False : 1

Every point of the interval [0, 1] is a cluster point of the interval (0, 1).

(b) Define uniform continuity of a function on a set. 1

(c) Find the cluster points of the set $A = \{1, 2\}$. 2

(d) State the sequential criterion for the limit of a function. 2

(e) If $\text{sgn}(x)$ denotes the signum function of x, then show that

$$\lim_{x \to 0} \text{sgn}(x)$$

does not exist. 2

(f) Using definition, show that

$$\lim_{x \to \infty} x^n = \infty \text{ for } n \in \mathbb{N}$$ 2

(g) State the discontinuity criterion for a function at a point. 2

(h) If $f: A \to \mathbb{R}$ and C is a cluster point of A, then prove that f can have only one limit at C. 3

(i) Use ε-δ definition to establish that

$$\lim_{x \to c} \frac{1}{x} = \frac{1}{c}$$ 3

(j) Let I be a closed bounded interval and let $f: I \to \mathbb{R}$ be continuous on I. Then prove that the set $f(I) = \{f(x) : x \in I\}$ is a closed bounded interval. 3

(k) State and prove preservation interval theorem. 1+3=4

(l) State the squeeze theorem. Applying this theorem, show that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$ 2+3=5

2. (a) State the interior extremum theorem. 1

(b) Show that the function $g: [-1, 1] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 1 & \text{for } 0 < x \le 1 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } -1 \le x < 0 \end{cases}$$

is not the derivative of any function on $[-1, 1]$. 2

(c) Let $I \subseteq \mathbb{R}$ be an interval and $f: I \to \mathbb{R}$ such that f has derivative at c. If $f'(c) > 0$, then show that there is a number $\delta > 0$ such that

$$f(x) > f(c), \ \forall x \in (c, c + \delta)$$ 3

(d) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b). Show that if

$$\lim_{x \to a} f'(x) = A$$

then $f'(a)$ exists and equals A. 3

Or

Let I be an interval and let $f : I \to \mathbb{R}$ be differentiable on I. Show that if the derivative f' is positive on I, then f is strictly increasing on I.

(e) Let f be a continuous function on the interval $I = [a, b]$ and c be an interior point of I. If f is differentiable on (a, c) and (c, b) and if there is a neighbourhood $(c - \delta, c + \delta) \subseteq I$ such that $f'(x) \geq 0$ for $c - \delta < x < c$ and $f'(x) \leq 0$ for $c < x < c + \delta$, then prove that f has relative maximum at c. 3

(f) Use the mean value theorem to show that

$$\frac{x-1}{x} < \ln x < x - 1 \text{ for } x > 1$$

4

(g) If $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 2x + \dfrac{1}{x^2}$ for $x \neq 0$, then find the points of relative extrema and the intervals on which the function is increasing. 4

(h) State and prove Rolle's theorem. 1+4=5

3. *(a)* What is the necessary condition for the existence of relative maximum of a differentiable function $f : I \to \mathbb{R}$ at a point $C \in I$? 1

(b) State True or False : 1

A convex function on an open interval is necessarily continuous.

(c) Prove that

$$1 - \frac{x^2}{2} \leq \cos x$$

for all $x \in \mathbb{R}$. 2

(d) Let $I \subseteq R$ be an open interval and let $f : I \to \mathbb{R}$ be differentiable on I and $f''(a)$ exists at $a \in I$. Show that

$$f''(a) = \lim_{h \to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

3

Or

If $f(x) = e^x$, then show that the remainder term in Taylor's theorem converges to zero as $x \to \infty$ for each fixed x_0 and x.

(e) Determine whether or not $x = 0$ is a point of relative extremum of the function

$$f(x) = \sin x - x$$ 3

(f) Let I be an open interval and let $f : I \to \mathbb{R}$ have a second derivative on I. Then prove that f is a convex function on I iff

$$f''(x) \geq 0 \quad \forall\, x \in I$$ 5

(g) State the Maclaurin's theorem. Stating the condition of validity of the expansion, expand $\cos x$ by using Maclaurin's series. 2+3=5

(h) State the Taylor's theorem and derive the Cauchy's form of remainder after n terms in Taylor's series. 2+3=5

Or

Let I be an interval and x_0 be an interior point of I. For $n \geq 2$, suppose that the derivatives $f', f'', \cdots, f^{(n)}$ exist and are continuous in a neighbourhood of x_0 and that $f'(x_0) = \cdots = f^{(n-1)}(x_0) = 0$, but $f^n(x_0) \neq 0$. Then prove that f has a relative minimum or maximum at x_0 according as $f^{(n)}(x_0) > 0$ or $f^{(n)}(x_0) < 0$, when n is even. Also show that if n is odd, then f has neither a relative maximum nor a relative minimum at x_0. 5

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Total No. of Printed Pages—7

3 SEM TDC MTMH (CBCS) C 6

2025

(Nov/Dec)

MATHEMATICS

(Core)

Paper : C–6

(Group Theory—I)

Full Marks : 80
Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* Find the inverse of the element

$$\begin{bmatrix} 2 & 6 \\ 3 & 5 \end{bmatrix}$$

in $GL(2, \mathbb{Z}_{11})$. 1

(b) State True or False : 1

The set {0, 1, 2, 3} is not a group under multiplication modulo 4.

(c) Prove that in a group G, there is only one identity element. 3

(d) Prove that a group G is Abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}\ \forall\ a, b \in G$. 4

(e) Describe the dihedral group D_4. Show that it is a group together with the operation composition. Is it an Abelian group? Justify. 6

Or

Describe the group of symmetries of a rectangle. Show that it is a group together with the operation composition. Is it Abelian? Justify.

2. *(a)* Give an example of a non-Abelian group having finite order. 1

(b) Find the order of all the elements of $U(10)$. 2

(c) Let G be an Abelian group with identity e. Show that $H = \{x \in G \mid x^2 = e\}$ is a subgroup of G. 3

(d) Define centre, $z(G)$ of a group G. Show that it is a subgroup of G. 1+3=4

(e) Let H be a non-empty finite subset of a group G and H is closed under the operation of G. Show that H is a subgroup of G. 5

Or

Let

$$G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \ a, b, c, d \in \mathbb{Z} \right\}$$

under addition. Let

$$H = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G \ \middle| \ a + b + c + d = 0 \right\}$$

Prove that H is a subgroup of G. What if 0 is replaced by 1?

3. (a) State True or False : 1

$U(8)$ is a cyclic group.

(b) Show that $\mathbb{Z}_{10} = \langle 3 \rangle$. 2

(c) Find all the generators of $\mathbb{Z}_6$. 2

(d) Let

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{bmatrix} \text{ and}$$

$$\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 4 & 3 & 5 \end{bmatrix}$$

Compute α^{-1}, $\alpha\beta$ and $\beta\alpha$. 3

(e) Let G be a finite group and let $a \in G$. Then prove that $a^{|G|} = e$. 3

(f) Let G be a group and H be a subgroup of G. Let $a \in G$. Prove that $aH = H$ if and only if $a \in H$. 4

(g) State and prove Lagrange's theorem for finite group. 1+4=5

Or

Show that every permutation of a finite set can be written as a cycle or as a product of disjoint cycles. 5

4. (a) Define normal subgroup. 1

(b) Find the order of each element in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. 2

(c) Prove that $SL(2, \mathbb{R})$ is a normal subgroup of $GL(2, \mathbb{R})$. 3

(d) Determine the number of elements of order 5 in $\mathbb{Z}_{25} \oplus \mathbb{Z}_5$. 4

(e) Let G and H be finite cyclic groups. Then prove that $G \oplus H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime. 5

Or

Let G be a group and let $z(G)$ be the centre of G. Prove that if $G|z(G)$ is cyclic, then G is Abelian.

5. *(a)* Define kernel of a homomorphism. 1

(b) State True or False : 1

The kernel of an isomorphism is the identity.

(c) Let ϕ be a homomorphism from a group G to a group $\overline{G}$ and let H be a subgroup of G. Then prove the following : 2+2=4

(i) $\phi(H)$ is a subgroup of $\overline{G}$

(ii) If H is Abelian, then $\phi(H)$ is Abelian

(d) Let ϕ be a group homomorphism from a group G to a group $\overline{G}$. Then show that $\ker\phi$ is a normal subgroup of G. 4

(e) Let ϕ be a group homomorphism from G to $\overline{G}$. Then prove that $G/\ker\phi \approx \phi(G)$. 5

Or

Let G be a group of permutations. For each σ in G, define

$$\operatorname{sgn}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is an even permutation} \\ -1 & \text{if } \sigma \text{ is an odd permutation} \end{cases}$$

Prove that sgn is a homomorphism from G to the multiplicative group $\{+1, -1\}$. What is the kernel? 1+4=5

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Total No. of Printed Pages—4

3 SEM TDC MTMH (CBCS) C 7

2025

(Nov/Dec)

MATHEMATICS

(Core)

Paper : C–7

(PDE and Systems of ODE)

Full Marks : 60
Pass Marks : 24

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* Find the degree of the equation

$$x\frac{\partial^2 z}{\partial x^2}+y\left(\frac{\partial z}{\partial y}\right)^{1/3}+kz=0$$ 1

(b) Write Lagrange's subsidiary equation of

$$xzp+yzq=xy$$ 1

(c) Find the complete solution of $p^2+q^2=m$. 1

(d) Form the differential equation of the set of all right circular cones whose axes coincide with z-axis. 5

Or

Solve $(x^2-yz)p+(y^2-zx)q=z^2-xy$.

(e) Show that the equations $xp - yq = x$ and $x^2 p + q = xz$ are compatible. 5

Or

Find the integral surface of $x^2 p + y^2 q + z^2 = 0$ which passes through the hyperbola $xy = x + y$, $z = 1$.

2. (a) Find Jacobi's auxiliary equation for 2

$$p_1 x_1 + p_2 x_2 - p_3^2 = 0$$

(b) Find complete integral of any *one* of the following : 4

(i) $q = (z + px)^2$

(ii) $pxy + pq + qy = yz$

(iii) $z^2 = pqxy$

(c) Find complete integral of 6

$$p_3 x_3 (p_1 + p_2) + x_1 + x_2 = 0$$

Or

Solve the boundary value problem $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ with $u(x, 0) = x^2(25 - x^2)$ by the method of separation of variables.

3. (a) Write Laplace equation. 1

(b) Classify the operator 2

$$t\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial x \partial t} + x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}$$

(c) Show that $u = f(x + y) + g(y - x)$ satisfies the equation $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$, where f and g are functions. 2

(d) Derive one-dimensional wave equation. 7

Or

Reduce the equation $\frac{\partial^2 z}{\partial x^2} + x^2 \frac{\partial^2 z}{\partial y^2} = 0$ to canonical form.

4. (a) Write one assumption of vibrating string problem. 1

(b) Fill in the blank : 1

The partial differential equation in case of vibrating string problem is formulated from the law of ______.

(c) Solve the one-dimensional wave equation by the method of separation of variables. 6

Or

Solve

$$\frac{\partial^2 u}{\partial x^2} = k^2 \left(\frac{\partial u}{\partial t}\right)$$

when $u(0, t) = u(l, t) = 0$, $u(x, 0) = \sin \frac{\pi x}{l}$.

5. (a) Write the equation $3\frac{d^2 x}{dt^2} + 6\frac{dx}{dt} - x = t^2$ in normal form. 1

(b) Transform the linear differential equation

$$\frac{d^3x}{dt^3}+2\frac{d^2x}{dt^2}-\frac{dx}{dt}-2x=e^{3t}$$

into system of first-order differential equation. 2

(c) Let $L \equiv D^2+2,\ f(t)=e^{2t}$, where $D \equiv \frac{d}{dt}$. Find $Lf(t)$. 2

(d) Describe the method of successive approximation. 4

Or

Compute $y(0{\cdot}2)$ for the differential equation $\frac{dy}{dx}=y^2-x^2$ with $y(0)=1$ using Euler's method.

(e) Using operator method, find the general solution of

$$\frac{dx}{dt}+\frac{dy}{dt}-2x-4y=e^t$$
$$\frac{dx}{dt}+\frac{dy}{dt}-y=e^{4t}$$

6

Or

Solve :

$$\frac{dx}{dt}=5x-2y$$
$$\frac{dy}{dt}=4x-y$$

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