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6 SEM TDC MTMH (CBCS) C 13

2 0 2 4

(May)

MATHEMATICS

(Core)

Paper : C-13

(Metric Spaces and Complex Analysis)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

*The figures in the margin indicate full marks
for the questions*

1. (a) Write the triangle inequality of metric space. 1
- (b) A metric d on a non-empty set may be negative. State True or False. 1
- (c) A metric space consists of two objects. Write that objects. 2
- (d) Define a pseudometric on a non-empty set. 2
- (e) Define a complete metric space. 2

(f) Answer any two from the following :
6×2=12

(i) Show that in any metric space X , each open sphere is an open set.

(ii) Let X be a metric space with metric d . Show that d_1 defined by

$$d_1 = \frac{d(x, y)}{1 + d(x, y)}$$

is also a metric on X .

(iii) Show that a Cauchy sequence is convergent if and only if it has a convergent subsequence.

(iv) Show that a subset of a metric space is bounded if and only if it is non-empty and is contained in some closed sphere.

2. (a) Write when a metric space is called sequentially compact. 1

(b) Write an example of an uniformly continuous function in a metric space. 1

(c) Define a continuous mapping in a metric space. 2

(d) Show that the homeomorphism on the set of all metric spaces is an equivalence relation. 5

Or

Let f is a continuous mapping of a metric space X into a metric space Y . Then show that if E is a connected subset of X , then $f(E)$ is connected.

(Continued)

(e) Let X and Y be metric spaces and f a mapping of X into Y . Then show that f is continuous at x_0 if and only if $x_n \rightarrow x_0 \Rightarrow f(x_n) \rightarrow f(x_0)$. 6

Or

Show that every compact metric space has the Bolzano-Weierstrass property.

3. (a) Define extended complex plane. 1

(b) If a function f is continuous throughout a region R , then it is not bounded. State True or False. 1

(c) Show that $\lim_{z \rightarrow 0} \frac{z}{\bar{z}}$ does not exist. 2

(d) Find the arg z , where $z = \frac{-5}{1+i\sqrt{2}}$. 2

(e) Show that $\frac{dw}{dz} = (\cos\theta - i\sin\theta) \frac{\partial w}{\partial r}$, $w = w(r, \theta)$ is an analytic function. 4

Or

Let $f(z) = z - \bar{z}$. Show that $f'(z)$ does not exist at any point.

(f) Describe the mapping $w = z^2$. 5

4. (a) Find the analytic function $f(z) = u + iv$, where $u(x, y) = \sinh x \sin y$. 5

(b) e^z may have negative value. State True or False. 1

- (c) Show that $\log(e^z) = z + 2n\pi i$, $n = 0, 1, 2, \dots$ 4

Or

Evaluate $\int_C \bar{z} dz$, where C is the right-

hand half of the circle $|z|=2$.

5. (a) If a series of complex numbers converges, then write to which the n th term converges as n tends to infinity. 1
- (b) Find the limit to which the sequence $z_n = \frac{1}{n^3} + i$, $n = 1, 2, \dots$ converges. 2
- (c) State and prove Liouville's theorem. 7

Or

Find the Taylor's series for the function

$$\frac{1}{(1+z^2)(z+2)}, \text{ when } |z| < 1.$$

6. (a) Define absolute convergence of a power series. 2
- (b) Define the circle of convergence of a power series. 2
- (c) Write when a power series is called uniformly convergent. 1
- (d) Find Laurent's series for the function

$$f(z) = \frac{4z+3}{z(z-3)(z+2)}$$

when $2 < |z| < 3$. 5

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2 0 2 4

(May)

MATHEMATICS

(Core)

Paper : C-14

(Ring Theory and Linear Algebra—II)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

*The figures in the margin indicate full marks
for the questions*

1. Answer any *three* from the following : $5 \times 3 = 15$

(a) Define polynomial ring and prove that if D is an integral domain, then $D[x]$ is also an integral domain.

(b) Let F be a field. Then prove that $F[x]$ is principal ideal domain.

(2)

(c) State division algorithm for $F[x]$ and find the quotient and remainder upon dividing $f(x) = 3x^4 + x^3 + 2x^2 + 1$ by $g(x) = x^2 + 4x + 2$ where $f(x)$ and $g(x)$ belong to $Z_5[x]$.

(d) State Eisenstein's criterion on irreducibility. And prove that, in an integral domain, every prime is irreducible.

2. Answer any three from the following : $5 \times 3 = 15$

(a) Prove that if F is a field, then $F[x]$ is Euclidean domain.

(b) Prove that every ideal of Euclidean domain is principal ideal.

(c) Prove that every principal ideal domain is unique factorization domain.

(d) Show that the ring

$$Z[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in Z\}$$

is an integral domain but not a unique factorization domain.

(3)

3. Answer any three from the following : $6 \times 3 = 18$

(a) Let V be an n -dimensional vector space over the field F and let $B = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a basis for V . Let $B' = \{f_1, f_2, \dots, f_n\}$ be the dual basis of B . Then prove that—

(i) for each linear functional f on V ,

$$f = \sum_{i=1}^n f(\alpha_i) f_i;$$

(ii) for each vector α in V , $\alpha = \sum_{i=1}^n f_i(\alpha) \alpha_i$.

(b) Find the dual basis of the basis set

$$B = \{(1, -2, 3), (1, -1, 1), (2, -4, 7)\}$$

of $V_3(R)$.

(c) Let W_1 and W_2 be subspaces of a finite dimensional vector space V . Then prove that—

$$(i) (W_1 + W_2)^\circ = W_1^\circ \cap W_2^\circ;$$

$$(ii) (W_1 \cap W_2)^\circ = W_1^\circ + W_2^\circ.$$

(d) Find the minimal polynomial of the real matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Also, show that minimal polynomial of a matrix or of a linear operator is unique.

(4)

4. (a) Show that the space generated by $(1, 1, 1)$ and $(1, 2, 1)$ is an invariant subspace of \mathbb{R}^3 under T , where

$$T(x, y, z) = (x+y-z, x+y, x+y-z) \quad 3$$

- (b) Prove that the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable over the field \mathbb{C} . 4

Or

Find all complex eigenvalues and eigenspaces of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

5. (a) If α and β are vectors in an inner product space $V(F)$ and $a, b \in F$, then prove that—

$$(i) \quad \|\alpha a + b\beta\|^2 = |a|^2 \|\alpha\|^2 + a\bar{b}(\alpha, \beta)$$

$$+ \bar{a}b(\beta, \alpha) + |b|^2 \|\beta\|^2;$$

$$(ii) \quad \operatorname{Re}(\alpha, \beta) = \frac{1}{4} \|\alpha + \beta\|^2 - \frac{1}{4} \|\alpha - \beta\|^2. \quad 5$$

- (b) If α and β are vectors in a real inner product space and if $\|\alpha\| = \|\beta\|$, then prove that $\alpha - \beta$ and $\alpha + \beta$ are orthogonal and interpret the result geometrically. 3+2=5

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(Continued)

(5)

- (c) Given the basis $(2, 0, 1)$, $(3, -1, 5)$ and $(0, 4, 2)$ for $V_3(\mathbb{R})$. Construct from it by the Gram-Schmidt process an orthonormal basis relative to the standard inner product space. 5

Or

Let V be a finite dimensional inner product space and let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be an orthonormal basis for V . Show that for any vectors $\alpha, \beta \in V$,

$$(\alpha, \beta) = \sum_{k=1}^n (\alpha, \alpha_k) \overline{(\beta, \alpha_k)}$$

6. (a) If T is skew, does it follow that so is T^2 ? What about T^3 ? 1+1=2

- (b) Answer any two from the following : 4×2=8

- (i) Let V be the vector space $V_2(\mathbb{C})$ with the standard inner product. Let T be the linear operator defined by

$$T(1, 0) = (1, -2), \quad T(0, 1) = (i, -1)$$

If $\alpha = (a, b)$, then find $T^* \alpha$.

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(Turn Over)

- (ii) Prove that a linear transformation E is an orthogonal projection if and only if $E = E^2 = E^*$.
- (iii) Prove that a necessary and sufficient condition that a self-adjoint linear transformation T on an inner product space V be \hat{O} is that $(T\alpha, \alpha) = 0, \forall \alpha \in V$.
