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6 SEM TDC MTMH (CBCS) C 13

2025

(May)

MATHEMATICS

(Core)

Paper : C-13

(Metric Space and Complex Analysis)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

*The figures in the margin indicate full marks
for the questions*

1. (a) Write the symmetric property of metric space. 1
- (b) Write when a subspace Y of a metric space will be completed. 1
- (c) Write when a metric is called a trivial metric. 2
- (d) Define an open set in a metric space. 2
- (e) Write when a metric space is called complete. 2

(f) Answer any two questions from the following : 6×2=12

(i) If (X, d) be a metric space and $x, y, z \in X$ be any three distinct points, then prove that

$$d(x, y) \geq |d(x, z) - d(z, y)|$$

(ii) Prove that in a metric space (X, d) each closed sphere is a closed set.

(iii) Prove that interior of a set is an open set.

2. (a) Write when a mapping from one metric space into another is said to be continuous. 1
- (b) Define a contracting mapping. 2
- (c) Define uniform continuity in a metric space. 2
- (d) Show that every contraction mapping is continuous. 4
- (e) Let X and Y be metric spaces and f be a mapping of X into Y . If f is a constant mapping, show that f is continuous. 6

Or

Prove that a subspace of the real line R is connected if and only if it is an interval.

3. (a) Write when a function of a complex variable is called a many-valued function. 1
- (b) Write the area of a parallelogram having sides z_1 and z_2 . 1

(c) Write the nature of the singularity of the function

$$f(z) = \frac{\sin z}{z} \quad 1$$

(d) Show that $|z_1 z_2| = |z_1| |z_2|$. 2

(e) Find the points where the function

$$f(z) = \frac{z}{z^2 + 1}$$

is not continuous. 2

(f) Show that $\sin^2 z + \cos^2 z = 1$. 3

(g) Prove the necessary condition for a function to be analytic. 5

Or

Find the image of the semi-infinite strip $x \geq 0, 0 \leq y \leq \pi$ under the transformation $w = e^z$.

4. (a) $e^z = 0$, for some complex number z . State true or false. 1
- (b) Show that $e^{2+3\pi i} = -e^2$. 2
- (c) Define a simply connected domain. 2
- (d) Evaluate $\int_0^{\pi/6} e^{i2t} dt$. 3

(e) Evaluate $\int_C \frac{z+2}{z} dz$, where C is the semi-circle $z = 2e^{i\theta}$ ($0 \leq \theta \leq \pi$). 4

Or

Find $\text{Im } f(z)$, where

$$\text{Re } f(z) = e^x (x \cos y - y \sin y)$$

of an analytic function $f(z)$.

5. (a) Write when the sequence $\{z_n\}$ converges. 1
 (b) If a series of complex numbers converges, then write to which n th term converges. 1
 (c) Find the limit of the sequence defined by $z_n = -2 + i \frac{(-1)^n}{n^2}$, $n = 1, 2, 3, \dots$ 3
 (d) Expand $f(z) = \log(1+z)$ in a Taylor series about $z=0$. 5

Or

Prove that the series $\sum_{n=1}^{n-1} \frac{z^{n-1}}{2^n}$ converges for $|z| < 2$.

6. (a) Write the statement of Laurent's theorem. 2
 (b) Find the Laurent series for

$$f(z) = \frac{z}{(z-1)(z-3)}$$

when $0 < |z-1| < 2$. 6

Or

Investigate the uniform convergence of the series $\sum_{n=0}^{\infty} (-1)^n (z^n + z^{n+1})$.

6 SEM TDC MTMH (CBCS) C 14

2025

(May)

MATHEMATICS

(Core)

Paper : C-14

(Ring Theory and Linear Algebra—II)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

*The figures in the margin indicate full marks
for the questions*

1. (a) If F is commutative, then write the condition such that $F[x]$ is invertible. 1
- (b) Prove that every Euclidean domain possesses unity. 2
- (c) Show that $x^2 + 3x + 2$ has four zeros in Z_6 . 2

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(d) Let F be a field. Then prove that the ring of polynomial $F[x]$ is principal ideal domain (PID). 4

(e) Prove that a polynomial of degree n over a field has at most n zeros, counting multiplicity. 6

Or

Let F be a field and let $f(x), g(x) \in F[x]$ with $g(x) \neq 0$. Then prove that there exist unique polynomials $q(x)$ and $r(x)$ in $F[x]$ such that $f(x) = g(x)q(x) + r(x)$ and either $r(x) = 0$ or $\deg r(x) < \deg g(x)$.

2. (a) What is the inverse of $1 + \sqrt{2}$ in $Z[\sqrt{2}]$? 1

(b) Define Euclidean domain. 1

(c) Test the irreducibility of the polynomial $x^5 + 9x^4 + 12x^2 + 6$ in Q . 2

(d) Prove that in a principal ideal domain, an element is irreducible if and only if it is a prime. 5

(e) Define unique factorization domain and prove that every field is unique factorization domain. 1+5=6

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(Continued)

(3)

Or

Prove that $Z[\sqrt{3}] = \{a + b\sqrt{3} \mid a, b \in Z\}$ is a Euclidean domain. 6

3. (a) Write when two linear functionals are said to be equal on a vector space $V(F)$. 1

(b) Define invariant subspace. 1

(c) If S_1 and S_2 are two subsets of a vector space $V(F)$ such that $S_1 \subseteq S_2$, then show that $S_2^\circ \subseteq S_1^\circ$. 2

(d) Prove that the subspace spanned by two subspaces each of which is invariant under some linear operator T , is itself invariant under T . 3

(e) Let V be an n -dimensional vector space over the field F and let

$$\beta = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$$

be a basis for V . Then prove that there is a uniquely determined basis

$$\beta' = \{f_1, f_2, \dots, f_n\}$$

for V' such that $f_i(\alpha_j) = \delta_{ij}$. 6

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(Turn Over)

Or

Let V be finite dimensional vector space over the field F and let W be a subspace of V . Then prove that

$$\dim W + \dim W^\circ = \dim V$$

4. (a) Write about the eigenvalues and eigenvectors of the identity matrix. 1
- (b) If V is n -dimensional vector space, then what is the condition that the linear operator T is diagonalizable? 1
- (c) Test the diagonalizability of the following matrix : 4

$$\begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

- (d) Define minimal polynomial and show that the minimal polynomial of the real matrix

$$\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

is $(x-1)(x-2)$.

1+5=6

Or

If $f(x)$ be the characteristic polynomial of T , then prove that $f(T) = \hat{0}$. 6

5. (a) Write the only vector that is orthogonal to itself. 1
- (b) Define orthogonal complement. 1
- (c) If α, β are vectors in an inner product space V , then prove that

$$\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\| \quad 4$$

Or

If W_1 and W_2 are subspaces of a finite dimensional inner product space, then prove that

$$(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$$

- (d) If $\beta = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$ is any finite orthonormal set in an inner product space V , and if β is any vector in V , then prove that

$$\sum_{i=1}^m |(\beta, \alpha_i)|^2 \leq \|\beta\|^2 \quad 6$$

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Or

In an inner product space, prove that

$$|(\alpha, \beta)| \leq \|\alpha\| \|\beta\|$$

6. (a) Write the two self-adjoint operators on any inner product space $V(F)$. 1
- (b) Define normal operator. 1
- (c) If T_1 and T_2 are normal operators on an inner product space with the property that either commutes with the adjoint of the other, then prove that $T_1 T_2$ is also normal operator. 2
- (d) Let V be the direct sum of its subspaces W_1 and W_2 . If E_1 is the projection on W_1 along W_2 , and E_2 is the projection on W_2 along W_1 , then prove that— 4
- (i) $E_1 + E_2 = I$;
- (ii) $E_1 E_2 = \hat{0}$, $E_2 E_1 = \hat{0}$.
- (e) If T_1 and T_2 are self-adjoint linear operators on an inner product space V , then prove that (i) $T_1 + T_2$ is self-adjoint and (ii) if $T_1 \neq \hat{0}$ and a is a non-zero scalar, then aT_1 is self-adjoint iff a is real. 5

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(7)

Or

Apply the Gram-Schmidt process to the vectors $(1, 0, 0)$, $(1, 1, 0)$, $(1, 1, 1)$ to obtain an orthonormal basis for $V_3(\mathbb{R})$ with the standard inner product.

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