3 SEM PG (CBCS) MTH C 7

2025

(December)

MATHEMATICS

Paper : MTH C 7

(Functional Analysis)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

UNIT–I

1. Answer ***any two*** of the following questions : 5×2=10

 (a) Define the norm of l^p space, $1 \le p < \infty$. Prove that l^p space is a Banach space.

(b) Let $(C[a,b], \|\cdot\|_\infty)$ be the Banach space. Take a fixed point $t_0 \in]a, b[$. Define the functional $f_{t_0} : X \to \mathbb{R}$ by $f_{t_0}(x) = x(t_0), x \in C[a,b]$. Then prove that f_{t_0} is a bounded linear functional with $\| f_{t_0} \| = 1$.

(c) Let $\|\cdot\|$ and $\|\cdot\|'$ be equivalent norms on a linear space X. Then show that $(X, \|\cdot\|)$ is a Banach space if and only if $(X, \|\cdot\|')$ is also Banach space.

2. Prove that : Dual space of l^1 is l^∞. 5

Or

In a finite dimensional normed space X, any subset $M \subset X$ is compact if and only if M is closed and bounded. (Prove)

UNIT–II

3. Answer ***any three*** of the following : 5×3=15

(a) State and prove closed graph theorem.

(b) State and prove Uniform Boundedness Principle.

(c) Let X and Y be normed spaces over the field $\mathbb{K}$ and $T : X \to Y$ a bounded linear operator then prove that : The mapping $T \to T^X$ is an isometric isomorphism of $B(X,Y)$ into $B(Y^*, X^*)$.

(d) Let X be a normed space over the field $\mathbb{K}$ and $x \in X$. Then prove that

$$\| x \| = sup\left\{\frac{| f(x) |}{\| f \|} : f \in X^*, f \neq 0\right\}.$$

(e) State and prove Hahn-Banach theorem for real normed space.

UNIT–III

4. Answer ***any three*** of the following questions : 5×3=15

(a) Let A be a convex subset of a Hilbert space H and let $\{x_n\}$ be a sequence in A such that :

$\lim_{n\to\infty} \|x_n\| = d = \inf_{x\in A} \|x\|$. Prove that $\{x_n\}$ converges in H.

(b) Let M be a closed subspace of a Hilbert space H. Then prove that $H = M \oplus M^{\perp}$.

(c) Let $\{e_\alpha\}_{\alpha\in\Lambda}$ be an orthonormal set in an inner product space X and $x \in X$. Then prove that

$\sum_{\alpha\in\Lambda} |< x, e_\alpha >|^2 \le \|x\|^2$ for every $x \in X$.

(d) Let H be a Hilbert space and H^* be its dual space. If $f \in H^*$ is an arbitrary but fixed functional, then show that there exists a unique vector $z \in H$ such that $f(x) = < x, z >, \forall x \in H$ where z depends on f and has the norm : $\|z\| = \|f\|$.

(e) Let H be a complex Hilbert space. Then show that every $T \in B(H)$ can be expressed uniquely as $T = A + iB$, where $A, B \in S(H)$.

UNIT–IV

5. (a) State and prove Banach fixed point theorem. 7

Or

State and prove Picard's existence and uniqueness theorem.

(b) Define multiplication and differential operator. Also, write their use in quantum mechanics. 3

(c) Give an application of Banach fixed point theorem to linear equation. Where $T : X \to X$ defined by

$$y = Tx = (x + b)$$

$$x = (\xi_1, \xi_2, \dots \xi_n); \; y = (\eta_1, \eta_2, \dots, \eta_n)$$

$C = (C_{jk})$ is a fixed real $n \times n$ matrix and $b \in X$ a fixed vector, with

$$d(x, y) = \max_j |\xi_i - \eta_j|.$$ 5

Or

Give an application of Banach fixed point theorem to Fredholm integral equation.

(All the symbols have their usual meaning).