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4 SEM TDC MTMH (CBCS) C 10

2026

(May/June)

MATHEMATICS

(Core)

Paper : C–10

(Ring Theory and Linear Algebra I)

Full Marks : 80
Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions.

1. *(a)* Define subring. Give example. 1+1=2

(b) Let a, b and c belong to a ring R. Then prove that

(i) $a0 = 0a = 0$

(ii) $a(-b) = (-a)b = -ab$ 2+2=4

Or

Prove that a ring R is without zero divisor if and only if for all $a, b, c \in R$, $ab = ac$ and $a \neq 0$ imply $b = c$. 4

(c) Show that a finite integral domain is a field. Give an example of the same. Also, give an example of an infinite integral domain which is not a field. 3+1+1=5

(d) Show that a field cannot have a non-trivial ideal. 4

(e) Prove that the intersection of two ideals of a ring R is again an ideal of R. Also show that $6Z$ is an ideal of $2Z$, where Z is the ring of integers. 3+2=5

Or

Define principal ideal domain and show that the set of integers Z is a principal ideal domain. 1+4=5

2. (a) If $f: R \to R'$ be a ring homomorphism, show that $f(0) = 0'$ where 0 and $0'$ are zero elements of R, R' respectively. 1+1=2

(b) Show that the mapping $\phi: x \to 5x$ from Z_4 to Z_{10} is a ring homomorphism. 3

(c) Prove that an integer n with decimal representation $a_k\, a_{k-1} \dots a_0$ is divisible by 9 if and only if $a_k + a_{k-1} + \dots + a_0$ is divisible by 9. 4

Or

Let ϕ be a ring homomorphism from a ring R to a ring S. Then prove that the mapping from $R/\ker\phi$ to $\phi(R)$ given by $r + \ker\phi \to \phi(r)$ is an isomorphism, i.e., $R/\ker\phi \cong \phi(R)$.

(d) State and prove second theorem of isomorphism of ring. 2+4=6

Or

Let D be an integral domain. Then prove that there exists a field F (field of quotients of D) that contains a subring isomorphic to D. 6

3. (a) Prove that the union of two subspaces of a vector space is a subspace of the vector space if and only if one of the subspaces is contained in the other. 5

(b) Show that for a subset W of a vector space V, $W = \text{span}(W)$ if and only if W is a subspace of V. 5

Or

Define basis of a vector space V. Show that the set $\beta = \{(1, 0, 1), (1, 1, 0), (0, 1, 1)\}$ is a basis of R^3. 1+4=5

(c) Let V be a vector space which is spanned by a finite set of vectors $\beta_1, \beta_2, \ldots, \beta_m$. Then prove that any independent set of vectors in V is finite and contain no more than m elements. 5

Or

If W_1 and W_2 are finite-dimensional subspaces of a vectorspace V, then show that $W_1 + W_2$ is a finite-dimensional subspace of V and

$$\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$$

4. (a) Define linear transformation. Show that the mapping $T: V_3(R) \to V_2(R)$ defined as

$$T(a_1, a_2, a_3) = (3a_1 - 2a_2 + a_3, a_1 - 3a_2 - 2a_3)$$

is a linear transformation from $V_3(R)$ into $V_2(R)$. 1+3=4

(b) Find the rank and nullity of the linear transformation $T: V_2(R) \to V_3(R)$ defined by $T(a, b) = (a+b, a-b, b)$. 4

(c) Let V and W be vector spaces over the field F. Then show that the set of all linear transformations from V into W is a vector space over the field F. 6

Or

Let V be a finite-dimensional vector space over the field F and let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an ordered basis for V. Let W be a vector space over the field F and let $\beta_1, \beta_2, \ldots, \beta_n$ be any vectors in W. Then prove that there is a unique linear transformation $T\alpha_j = \beta_j$, $j = 1, 2, \ldots, n$.

5. (a) Define isomorphism of vector spaces. Prove that every n-dimensional vector space over the field F is isomorphic to the vector space F^n. 1+4=5

(b) Let V be an n-dimensional vector space over the field F and W an m-dimensional vector space over F. Let β be an ordered basis for V and β' an ordered basis for W. Prove that for each linear transformation T from V into W, there is an $m \times n$ matrix A with entries in F such that $[T_\alpha]_{\beta'} = A[\alpha]_\beta$. 5

Or

Let V be the vector space of all polynomials of degree three or less over R. Let $\beta = \{f_1, f_2, f_3, f_4\}$ be the basis of V, where $f_j(x) = x^{j-1}$. Then find the matrix representation of the usual differential operator D on V in the ordered basis β.

(c) Let T be the linear operator on R^3 defined by

$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$$

(i) Find the matrix of T in the standard ordered basis of R^3.

(ii) If the transition matrix P from the standard ordered basis β to the ordered basis $\beta' = \{(1, 0, 1), (-1, 2, 1), (2, 1, 1)\}$ is

$$P = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

find the matrix of T relative to the ordered basis β'. 3+3=6

Or

Define similar matrix. Prove that the matrix

$$[T]_\beta = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

of the linear operator T on R^2 relative to standard ordered basis of R^2 defined by $T(x_1, x_2) = (x_1, 0)$ is similar to the matrix

$$[T]_{\beta'} = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}$$

of the linear operator T with the ordered basis $\beta = \{(1, 1), (2, 1)\}$. 1+5=6

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